The Relation Between the Two Ways to Deduce the Relaxation and Retardation Spectra

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Synopsis

There are two different ways to deduce the relaxation and retardation spectra. The first way is from a generalized Maxwell model and Voigt model. The second way is from the Boltzmann superposition principle. Up to now the connection between the two ways has not been studied. Some problems in the first way have not been treated properly. This paper will solve the problems and discuss the relation between them from a mathematical and mechanical viewpoint.

INTRODUCTION

In polymer mechanics and viscoelastic mechanics, there are two ways to deduce the relaxation and retardation spectra.[1-6] The first way, which we can find in almost all textbooks of polymer mechanics, starts from a generalized Maxwell model (see Fig. 1). From the model we obtain the stress response after a constant strain is applied on the model:

$$\sigma(t) = \epsilon_0 \sum_{i=1}^{n} G_i e^{-t/\tau_i} \tag{1}$$

Then, we have the relaxation function

$$G(t) = G_0 + \sum_{i=1}^{n} G_i e^{-t/\tau_i} \tag{2}$$

If the number of Maxwell elements increases infinitely, G_i is replaced by $F(\tau_i)\, d\tau_i$. The relaxation function becomes

$$G(t) = G_0 - \int_0^\infty F(\tau) e^{-t/\tau}\, d\tau \tag{3}$$

It is more common to use logarithmic time scale. Then, we have

$$G(t) = G_0 + \int_{-\infty}^{\infty} H(\tau) e^{-t/\tau}\, d \ln \tau \tag{4}$$

where $H(\tau)$ is usually called the relaxation spectrum. In Ref. 6, $F(\tau)$ is also called the relaxation spectrum. To distinguish the two different spectra, we call $F(\tau)$ the linear relaxation spectrum and $H(\tau)$ the relaxation spectrum or

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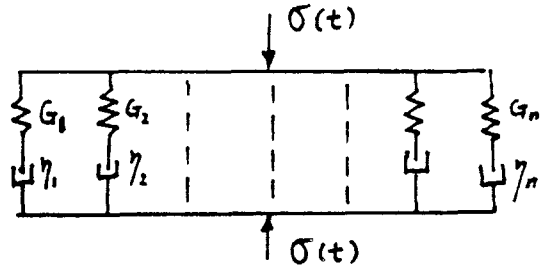

Fig. 1. The generalized Maxwell model.

logarithmic relaxation spectrum. The relation between them is

$$H(\tau) = \tau F(\tau) \tag{5}$$

On the other hand, if the generalized Voigt model (Fig. 2) is applied a constant stress, we have the strain:

$$\epsilon(t) = \sigma_0 \sum_{i=1}^{n} J_i(1 - e^{-t/\tau_i}) \tag{6}$$

In a similar way, we have the creep function

$$J(t) = J_0 + \int_0^\infty I(t)(1 - e^{-t/\tau})\, d\tau + t/\eta_0 \tag{7}$$

We call $I(\tau)$ the linear retardation spectrum. Changing the linear time scale to logarithmic scale, we obtain

$$J(t) = J_0 + \int_{-\infty}^{\infty} L(\tau)(1 - e^{-t/\tau})\, d \ln \tau + t/\eta_0 \tag{8}$$

where $L(\tau)$ is called retardation spectrum or logarithmic retardation spectrum. J_0 and t/η_0 are added to represent the instantaneous component of strain and the viscous flow. The relation between $I(\tau)$ and $L(\tau)$ is

$$L(\tau) = \tau I(\tau) \tag{9}$$

The second way to obtain the spectra is directly from the Boltzmann superposition principle.[6] The results are the same with eqs. (3) and (7).

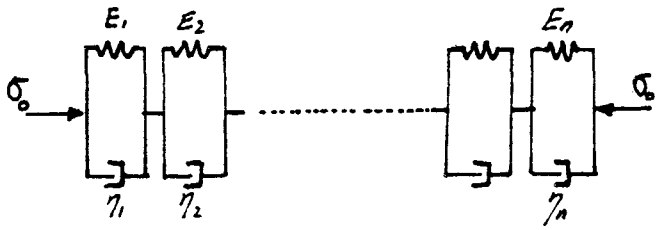

Fig. 2. The generalized Voigt model.

When the linear relaxation spectrum is discrete, the spectrum has a form like[6]

$$F(\tau) = \sum_{i=1}^{n} G_i \delta(\tau - \tau_i) \tag{10}$$

From eq. (5)

$$\dot{H}(\tau) = \tau \sum_{i=1}^{n} G_i \delta(\tau - \tau_i) \tag{11}$$

Insert eq. (10) into eq. (3) to obtain the relaxation function

$$G(t) = G_0 + \sum_{i=1}^{n} G_i e^{-t/\tau_i}$$

which is the same as eq. (2).

In the analogous method, the discrete retardation spectrum is

$$I(\tau) = \sum_{i=1}^{n} J_i \delta(\tau - \tau_i) \tag{12}$$

After substitution of $I(\tau)$ into eq. (7), we have results like eq. (6). According to eq. (9), the retardation spectrum is

$$L(\tau) = \tau \sum_{i=1}^{n} J_i \delta(\tau - \tau_i) \tag{13}$$

There are some problems rising from the first way. The deduction by the first way is based on the replacement of G_i in eq. (2) by $F(\tau_i)\, d\tau_i$. How does a quantity G_i relate to a differential $F(\tau_i)\, d\tau_i$? The relation between them is not clear. The mathematical and mechanical meaning of the replacement is not described properly. Furthermore, what is the relation between the two ways? Can the two ways be transversed to each other? The following section will discuss these questions.

THE PROCEDURE OF DEDUCTION BY THE FIRST WAY

Go back to the generalized Maxwell model to which a constant strain ϵ_0 is applied. Then the stress is

$$\sigma_i(t) = G_i e^{-t/\tau_i} \epsilon_0 \tag{14}$$

$$\sigma(t) = \sum_{i=1}^{n} G_i e^{-t/\tau_i} \epsilon_0 \tag{15}$$

where $\sigma_i(t)$ is the stress in the ith element and $\sigma(t)$ is the total stress on the model.

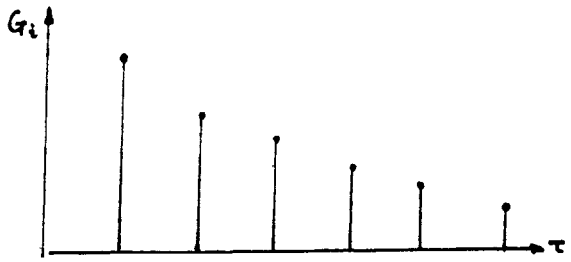

Fig. 3-1. The distribution of G_i in n-element M-model.

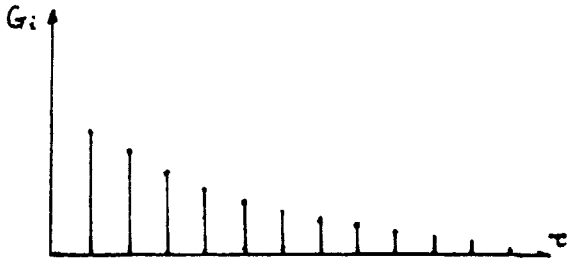

Fig. 3-2. The distribution of G_i in $2n$-element M-model.

Here we suppose the distribution of G_i of the n-element Maxwell model is given in Figure 3(1). According to eq. (14), we can calculate each $\sigma_i(t)$ and plot them in Figure 4(1). From eq. (15) we obtain $\sigma(t)$ and show it in Figure 5(1), where the value of $\sigma(t)$ at τ_i, $\tau_i < \tau_n$, is $\sum_{j=1}^{i}\sigma_j(t)$.

If we add more elements in the model, for example, the $2n$-element model, and keep the strain ϵ_0 and $\sigma(t)$ invariable, we can plot the G_i, $\sigma_i(t)$ and $\sigma(t)$ in Figures 3(2), 4(2), and 5(2), respectively. It is apparent that the magnitudes

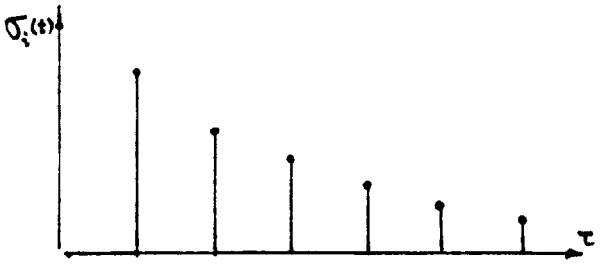

Fig. 4-1. The distribution of $\sigma_i(t)$ in n-element M-model.

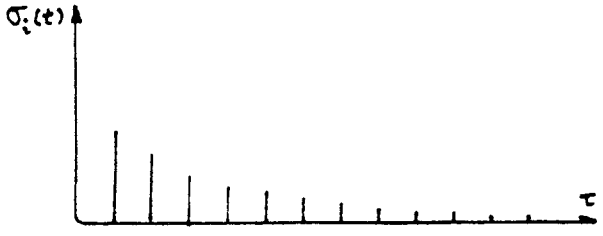

Fig. 4-2. The distribution of $\sigma_i(t)$ in $2n$-element M-model.

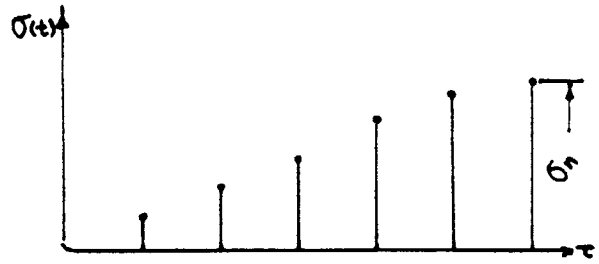

Fig. 5-1. The values of $\sigma(t)$ in n-element M-model.

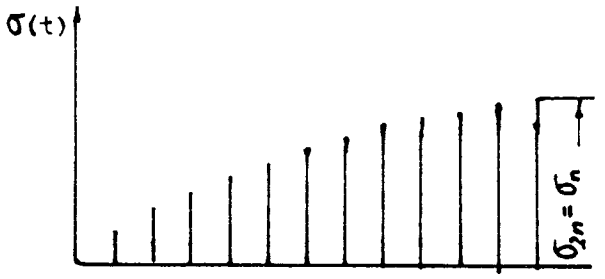

Fig. 5-2. The values of $\sigma(t)$ in $2n$-element M-model.

of G_i and $\sigma_i(t)$ will decrease if the number of elements increases. If the number of elements n is infinite, G_i and $\sigma_i(t)$ will approach infinitesimals. This is an important result. We also can obtain such a result by Cauchy's method to decide convergent series. If the series in eq. (15) is convergent, i.e.,

$$\lim_{n\to\infty} \sum_{i=1}^{n} G_i e^{-t/\tau_i} = A \tag{16}$$

The convergent series should have the following property

$$\lim_{i\to\infty} \left[G_i e^{-t/\tau_i}\right]^{1/i} < 1 \tag{17}$$

We know

$$\lim_{i\to\infty} e^{-t/i\tau_i} = 1 \tag{18}$$

Therefore,

$$\left[G_i e^{-t/\tau_i}\right]^{1/i} \leq G_i^{1/i} \tag{19}$$

If $\lim_{i\to\infty} G_i^{1/i} < 1$, the series in eq. (16) must converge. Then G_i must approach zero when $i \to \infty$.

Based on the above discussion, we use the symbol ΔG_i for G_i and $\Delta\sigma_i(t)$ for $\sigma_i(t)$ when $n \to \infty$. $n \to \infty$ means the distribution of G_i approaches a continuous distribution, i.e., the G_i's are very, very close to each other. Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} G_i e^{-t/\tau_i} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta G_i\, e^{-t/\tau_i} \tag{20}$$

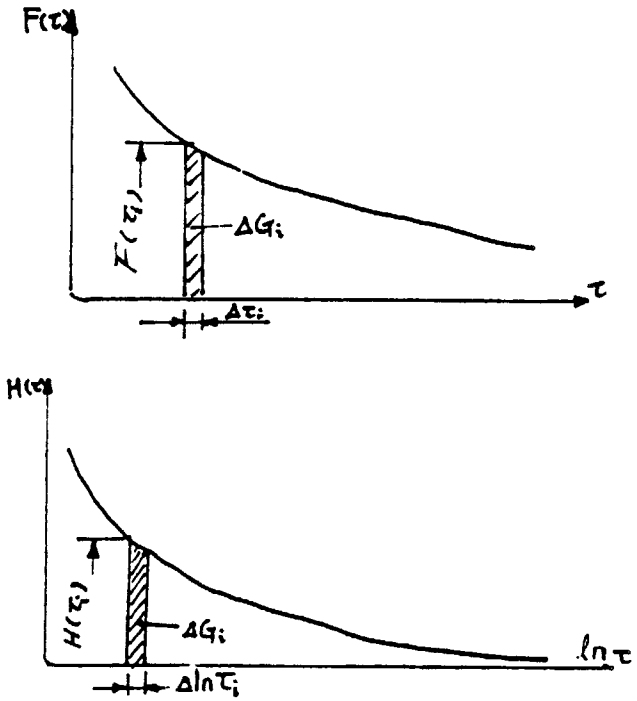

Fig. 6. The mechanical meanings of $F(\tau)$ and $H(\tau)$.

and the infinitesimal ΔG_i can be expressed by

$$\Delta G_i = F(\tau_i)\,\Delta\tau_i = H(\tau_i)\Delta \ln \tau_i \tag{21}$$

$F(\tau_i)$ is the density of ΔG_i in the infinitesimal interval $\Delta\tau_i$, which includes τ_i, and $H(\tau_i)$ is the density of ΔG_i in the logarithmic time interval $\Delta \ln \tau_i$. The mechanical meanings of the two relaxation spectra are shown in Figure 6, where the shaded area is ΔG_i. When $\Delta\tau_i \to 0$, $\Delta G_i \to 0$. This agrees with the property of ΔG_i, i.e., $n \to \infty$, $\Delta G_i \to 0$. By now we have described the nature of the replacement of G_i by $F(\tau_i)\,d\tau_i$. The mathematical and mechanical background of the first way has been explained in detail.

The analogous method can be used to prove the substitution of $I(\tau_i)\,d\tau_i$ for J_i when the number of Voigt elements increases infinitely.

THE RELATION BETWEEN THE TWO WAYS

As we know, the second way starts from the superposition principle and produces the same results as eqs. (3) and (7). For a discrete linear relaxation spectrum,[6]

$$F(\tau) = \sum_{i=1}^{n} G_i \delta(\tau - \tau_i) \tag{10}$$

where $\delta(\tau - \tau_i)$ is the Dirac delta function. This discrete spectrum does not come from the generalized Maxwell model. It is derived from the mathematic properties of Dirac function.

When $n \to \infty$, i.e., the G_i is getting closer and closer, as we discussed above, $G_i \to \Delta G_i = F(\tau_i)\, d\tau_i = H(\tau_i)\Delta \ln \tau_i$.

Then,

$$F(\tau) = \lim_{n\to\infty} \sum_{i=1}^{n} F(\tau_i)\delta(\tau - \tau_i)\,\Delta\tau_i = \lim_{n\to\infty} \sum_{i=1}^{n} H(\tau_i)\delta(\tau - \tau_i)\Delta \ln \tau_i \quad (22)$$

This series is the integral

$$F(\tau) = \int_0^\infty F(\tau_i)\delta(\tau - \tau_i)\, d\tau_i \quad (23)$$

According to the properties of the Dirac function, the right side of eq. (23) is really equal to $F(\tau)$. Here, we use relation (21) from the first way to transform the discrete spectrum (obtained from the second way) into the continuous spectrum when $n \to \infty$. The results show that the two ways are identical to each other and can be transversed from one to the other. This relation is also available for the retardation spectrum.

CONCLUSION

From the mechanical properties of the generalized Maxwell model and the generalized Voigt model, we can deduce the relaxation and retardation spectra. From the Boltzmann superposition principle we can obtain the spectra too and use two special Dirac function series to represent the discrete spectra. Those are the two ways to deduce the spectra.

The author studies the first way in detail and explains the mathematic and mechanical meanings of its deduction clearly. When the number of elements in those models increases, the magnitudes of G_i and J_i decrease, and G_i and J_i approach infinitesimals when $n \to \infty$. This makes the replacement of G_i by $F(\tau_i)\, d\tau_i$ or $H(\tau_i)\, d \ln \tau_i$ and J_i by $I(\tau_i)\, d\tau_i$ or $L(\tau_i)\, d \ln \tau_i$ to be realized. Using these relations in the second way, even though the discrete spectra (10) and (12) do not come from the first way, we can obtain the same continuous spectra from the discrete spectra (10) and (12) when $n \to \infty$. This study connects the two ways and proves the identity between the two ways.

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